Chapter 1

Linearni diferencialni rovnice

Postup:

(A) Nalezeni obecneho homogenniho reseni $y_h(t)$. Nebo tez prostoru reseni homogenni rovnice, ci baze tohoto prostoru.

(B1) Nalezeni partikularniho reseni $y_p(t)$ pomoci metody specialni prave strany.

(B2) Nalezeni partikularniho reseni $y_p(t)$ pomoci metody variace konstant.

(C) Nalezeni obecneho reseni $y(t) = y_h(t) + y_p(t)$. V pripade pocatecni podminky, nalezeni reseni splnujiciho pocatecni podminku.

1.1 Homogenni rovnice

Postup:

(i) Nalezeni charakteristickeho polynomu $\chi(t)$.

(ii) Nalzeni korenu charakteristickeho polynomu $\chi(t)$ splolu s nasobnosti techto korenu.

(iii) Nalezeni baze prostoru reseni homogenni rovnice pomoci vety o tvaru fundamentálního systému (V13). Prvky baze maji tvar $e^{\alpha t}$, kde α je koren charakteristickeho polynomu. V pripade vycenasobneho korenu se pak jeste prenasobuje t-ckem (pripadne vyssi mocninou t-cka).

(iv) Zapsani $y_h(t)$ jakozto linearni kombinace fundamentalniho systemu z bodu (iii). V pripade pocatecnich podminek, nalezeni reseni vyhovujiciho pocatecnim podminkam. Definicni obor reseni je $\mathbb{R}$.

1.1.1 $y'' + 4y + 4y = 0$

(i) $\chi(t) = t^2 + 4t + 4$,

(ii) $\{-2, -2\}$,

(iii) $\{e^{-2t}, te^{-2t}\}$,

(iv) $y(t) = ae^{-2t} + bte^{-2t}$, $a, b \in \mathbb{R}$, $t \in \mathbb{R}$.

1.1.2 $y'' - 3y' + 2y = 0$

(i) $\chi(t) = t^2 - 3t + 2$,

(ii) $\{1, 2\}$,

(iii) $\{e^t, e^{2t}\}$,

(iv) $y(t) = ae^t + be^{2t}$, $a, b \in \mathbb{R}$, $t \in \mathbb{R}$.

1.1.3 $y'' - 6y' + 13y = 0$

(i) $\chi(t) = t^2 - 6t + 13$,

(ii) $\{3 \pm 2i\}$,

(iii) $\left\{e^{3t}\cos(2t), e^{3t}\sin(2t)\right\}$,

(iv) $y(t) = e^{3t}(a\cos(2t) + b\sin(2t))$, $a, b \in \mathbb{R}$, $t \in \mathbb{R}$.

1.1.4 $y^{(4)} + 6y'' + 9y = 0$

(i) $\chi(t) = t^4 + 6t^2 + 9$,

(ii) $\{\pm i\sqrt{3}, \pm i\sqrt{3}\}$,

(iii) $\left\{\cos(\sqrt{3}t), \sin(\sqrt{3}t), t\cos(\sqrt{3}t), t\sin(\sqrt{3}t)\right\}$,

(iv) $y(t) = \cos(\sqrt{3}t)(a + ct) + \sin(\sqrt{3}t)(b + dt)$, $a, b, c, d \in \mathbb{R}$, $t \in \mathbb{R}$.

1.1.5 $y^{(6)} - 2y^{(3)} + 2y = 0$

(i) $\chi(t) = t^6 - 2t^3 + 2$,

(ii) $\left\{\sqrt[6]{2}(\cos(\alpha) + i\sin(\alpha));\ \alpha \in \left\{\frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}\right\}\right\}$,

(iii) $\left\{e^{\sqrt[6]{2}\cos(\alpha)t}\cos(\sqrt[6]{2}\sin(\alpha)t), e^{\sqrt[6]{2}\cos(\alpha)t}\sin(\sqrt[6]{2}\sin(\alpha)t);\ \alpha \in \left\{\frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}\right\}\right\}$,

(iv) $y(t) = e^{\sqrt[6]{2}\cos(\alpha)t}\sum_{i=1}^{3}(a_i\cos(\sqrt[6]{2}\sin(\alpha_i)t) + b_i\sin(\sqrt[6]{2}\sin(\alpha_i)t))$, $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$, $\alpha_1 = \frac{\pi}{12}$, $\alpha_2 = \frac{7\pi}{12}$, $\alpha_3 = \frac{3\pi}{4}$, $t \in \mathbb{R}$.

1.2 Rovnice se specialni pravou stranou

Postup nalezeni partikularniho a obecneho reseni:

(v) Nalezeni m, μ, ν a $k = \max\{stP, stQ\}$, z vety o specialni prave strane (V14), kde prava strana je rovna $e^{\mu t}(P(t)\cos(\nu t) + Q(t)\sin(\nu t))$. Cislo m udava, jakou nasobnost ma cislo $\mu + i\nu$ jakozto korene chrakteristickeho polynomu.

(vi) Pomoci k vyjadrime obecne tvary polynomu R, S (napr.: $k = 2$ implikuje $R(t) = at^2 + bt + c$, kde $a, b, c \in \mathbb{R}$. Dosadime tyto obecne polynomy a drive nalezene m, μ, ν do vety (V14) a obdrzime obecny tvar partikularniho reseni $y_p(t) = t^m e^{\mu t}(R(t)\cos(\nu t) + S(t)\sin(\nu t))$, $t \in \mathbb{R}$.

(vii) Dosadime partikularni reseni do rovnice a dopocitame presny tvar polynomu R a S a tim tez presny tvar $y_p(t)$.

(xiii) Nalezneme obecne reseni viz bod (C). V pripade pocatecni podminky, nalezeni reseni splnujiciho pocatecni podminku.

1.2.1 $y'' - 2y' - 3y = e^{4t}$

(i) $\chi(t) = t^2 - 2t - 3$,

(ii) $\{3, -1\}$,

(iii) $\{e^{-t}, e^{3t}\}$,

(iv) $y_h(t) = Ae^{-t} + Be^{3t}$, $A, B \in \mathbb{R}$, $t \in \mathbb{R}$.

(v) $m = 0, \mu = 4, \nu = 0, k = 0$.

(vi) $R(t) = a$, $y_p(t) = ae^{4t}$, kde $a \in \mathbb{R}$, $t \in \mathbb{R}$.

(vii) $a = \frac{1}{5}$, $y_p(t) = \frac{1}{5}e^{4t}$, $t \in \mathbb{R}$.

(xiii) $y(t) = \frac{1}{5}e^{4t} + Ae^{-t} + Be^{3t}$, $A, B \in \mathbb{R}$, $t \in \mathbb{R}$.

1.2.2 $y''(t) - 2y'(t) + 5y(t) = \cos(t)$

(i) $\chi(t) = t^2 - 2t + 5$,

(ii) $\{1 \pm 2i\}$,

(iii) $\{e^t \cos(2t), e^t \sin(2t)\}$,

(iv) $y_h(t) = e^t(A\cos(2t) + B\sin(2t))$, $A, B \in \mathbb{R}$, $t \in \mathbb{R}$.

(v) $m = 0, \mu = 0, \nu = 1, k = 0$.

(vi) $R(t) = a$, $S(t) = b$, $y_p(t) = a\cos(t) + b\sin(t)$, kde $a, b \in \mathbb{R}$, $t \in \mathbb{R}$.

(vii) $a = \frac{1}{5}$, $b = -\frac{1}{10}$, $y_p(t) = \frac{1}{5}\cos(t) - \frac{1}{10}\sin(t)$, $t \in \mathbb{R}$.

(xiii) $y(t) = \frac{1}{5}\cos(t) - \frac{1}{10}\sin(t) + e^t(A\cos(2t) + B\sin(2t))$, $A, B \in \mathbb{R}$, $t \in \mathbb{R}$.

1.2.3 $y'''(t) + y''(t) = t$

(i) $\chi(t) = t^3 + t^2$,

(ii) $\{0, 0, -1\}$,

(iii) $\{1, t, e^{-t}\}$,

(iv) $y_h(t) = A + Bt + Ce^{-t}$, $A, B, C \in \mathbb{R}$, $t \in \mathbb{R}$.

(v) $m = 2, \mu = 0, \nu = 0, k = 1$.

(vi) $R(t) = at + b$, $y_p(t) = at^3 + bt^2$, kde $a, b \in \mathbb{R}$, $t \in \mathbb{R}$.

(vii) $a = \frac{1}{6}$, $b = -\frac{1}{2}$, $y_p(t) = \frac{1}{6}t^3 - \frac{1}{2}t^2$, $t \in \mathbb{R}$.

(xiii) $y(t) = \frac{1}{6}t^3 - \frac{1}{2}t^2 + A + Bt + Ce^{-t}$, $A, B, C \in \mathbb{R}$, $t \in \mathbb{R}$.

1.3 Rovnice s pravou stranou ve tvaru souctu specialnich pravych stran

V nekterych pripadech nema prava strana specialni tvar, ale ma tvar souctu vice specialnich pravych stran. Tedy, $PS = \sum_{i=1}^{s} f_i(t)$, kde $f_i(t)$ ma specialni tvar (viz veta V14) pro $i = 1, \ldots, s$. V takovych pripade krome homogenniho reseni $y_h(t)$ spocitame s partikularnich reseni $y_p^i(t)$, $i = 1, \ldots, s$, ktera budou odpovidat prislusnym pravym stranam. Obecne reseni pak bude mit tvar $y(t) = y_h(t) + \sum_{i=1}^{s} y_p^i(t)$, $t \in \mathbb{R}$.

1.3.1 $y''' - y'' - 2y' = e^{2t} + t^3 + 3t^2 + 1$

(i) $\chi(t) = t^3 - t^2 - 2t$,

(ii) $\{-1, 0, 2\}$,

(iii) $\left\{e^{-t}, 1, e^{2t}\right\}$,

(iv) $y_h(t) = Ae^{-t} + B + Ce^{2t},\ A, B, C \in \mathbb{R},\ t \in \mathbb{R}$.

$f_1(t) = e^{2t}$:

$(\text{v})_1$ $m = 1, \mu = 2, \nu = 0, k = 0$.

$(\text{vi})_1$ $R(t) = a,\ y_p^1(t) = ate^{2t}$, kde $a \in \mathbb{R},\ t \in \mathbb{R}$.

$(\text{vii})_1$ $a = \frac{1}{6}$.

$f_2(t) = t^3 + 3t^2 + 1$:

$(\text{v})_2$ $m = 1, \mu = 0, \nu = 0, k = 3$.

$(\text{vi})_2$ $R(t) = at^3 + bt^2 + ct + d,\ y_p^2(t) = at^4 + bt^3 + ct^2 + dt$, kde $a, b, c, d \in \mathbb{R},\ t \in \mathbb{R}$.

$(\text{vii})_2$ $a = -\frac{1}{8},\ b = -\frac{1}{4},\ c = -\frac{3}{8},\ d = -\frac{7}{8}$.

(xiii) $y(t) = y_h(t) + y_p^1(t) + y_p^2(t) = Ae^{-t} + B + Ce^{2t} + \frac{1}{6}te^{2t} - \frac{1}{8}t(t^3 + 2t^2 + 3t + 7)$, $A, B, C \in \mathbb{R},\ t \in \mathbb{R}$.

1.3.2 $y'' + 3y' + 2y = \sin(t) + \sin(2t)$

(i) $\chi(t) = t^2 + 3t + 2$,

(ii) $\{-2, -1\}$,

(iii) $\left\{e^{-2t}, e^{-t}\right\}$,

(iv) $y_h(t) = Ae^{-2t} + Be^{-t},\ A, B \in \mathbb{R},\ t \in \mathbb{R}$.

$f_1(t) = \sin(t)$:

$(\text{v})_1$ $m = 0, \mu = 0, \nu = 1, k = 0$.

$(\text{vi})_1$ $R(t) = a,\ S(t) = b,\ y_p^1(t) = a\cos(t) + b\sin(t)$, kde $a, b \in \mathbb{R},\ t \in \mathbb{R}$.

$(\text{vii})_1$ $a = -\frac{3}{10},\ b = \frac{1}{10}$.

$f_2(t) = \sin(2t)$:

$(\text{v})_2$ $m = 0, \mu = 0, \nu = 2, k = 0$.

$(\text{vi})_2$ $R(t) = a,\ S(t) = b,\ y_p^2(t) = a\cos(2t) + b\sin(2t)$, kde $a, b \in \mathbb{R},\ t \in \mathbb{R}$.

$(\text{vii})_2$ $a = -\frac{3}{20},\ b = -\frac{1}{20}$.

(xiii) $y(t) = y_h(t) + y_p^1(t) + y_p^2(t) = Ae^{-2t} + Be^{-t} + \frac{1}{20}(2\sin(t) - 6\cos(t) - \sin(2t) - 3\cos(2t))$, $A, B \in \mathbb{R},\ t \in \mathbb{R}$.

1.3.3 $4y''' + y' = 3e^t + 2\sin\left(\frac{t}{2}\right)$

(i) $\chi(t) = 4t^3 + t$,

(ii) $\left\{0, \pm\frac{1}{2}i\right\}$,

(iii) $\left\{1, \cos\left(\frac{t}{2}\right), \sin\left(\frac{t}{2}\right)\right\}$,

(iv) $y_h(t) = A + B\cos\left(\frac{t}{2}\right) + C\sin\left(\frac{t}{2}\right)$, $A, B, C \in \mathbb{R}$, $t \in \mathbb{R}$.

$f_1(t) = 3e^t$:

(v)$_1$ $m = 0, \mu = 1, \nu = 0, k = 0$.

(vi)$_1$ $R(t) = a$, $y_p^1(t) = ae^t$, kde $a \in \mathbb{R}$, $t \in \mathbb{R}$.

(vii)$_1$ $a = \frac{3}{5}$.

$f_2(t) = 2\sin\left(\frac{t}{2}\right)$:

(v)$_2$ $m = 1, \mu = 0, \nu = \frac{1}{2}, k = 0$.

(vi)$_2$ $R(t) = a$, $S(t) = b$, $y_p^2(t) = at\cos\left(\frac{t}{2}\right) + bt\sin\left(\frac{t}{2}\right)$, kde $a, b \in \mathbb{R}$, $t \in \mathbb{R}$.

(vii)$_2$ $a = 0$, $b = -1$.

(xiii) $y(t) = y_h(t) + y_p^1(t) + y_p^2(t) = A + B\cos\left(\frac{t}{2}\right) + C\sin\left(\frac{t}{2}\right) + \frac{3}{5}e^t - t\sin\left(\frac{t}{2}\right)$, $A, B, C \in \mathbb{R}$, $t \in \mathbb{R}$.

1.4 Rovnice s pravou stranou

Necht funkce $y_1, \dots, y_n$ tvori fundamentalni system prislusne homogenni rovnice. Postup nalezeni partikularniho a obecneho reseni pomoci metody Variace konstant (V11):

(viii) Urcime definicni obor prave strany D_{PS}.

(ix) Sestavime soustavu n rovnic z vety V11 pro nezname $c_1', \dots, c_n'$, kde i-ta rovnice ma tvar $\sum_{j=1}^n c_j' y_j^{(i-1)} = 0$ pro $i = 1, \dots, n-1$ a n-ta rovnice ma tvar $\sum_{j=1}^n c_j' y_j^{(n-1)} = PS$.

(x) Vyresime soustavu rovnic z predchoziho kroku.

(xi) Nalezneme $c_1, \dots, c_n$, kde $c_i(t) = \int c_i'(t)dt$.

(xii) Pouzijeme V11 na nalezeni partikularniho reseni ve tvaru $y_p(t) = \sum_{j=1}^n c_j(t)y_j(t)$, $t \in I$, kde I je nejaky maximalni podinterval D_{PS}.

(xiii) Nalezneme obecne reseni viz bod (C). V pripade pocatecni podminky, nalezeni reseni splnujiciho pocatecni podminku.

Tento zpusob je vetsinou zdlouhavejsi nez metoda specialni prave strany, a tedy pokud je prava strana ve specialnim tvaru, tak je vhodnejsi nepouzivat tuto metodu.

1.4.1 $y'' - y = \frac{e^t - e^{-t}}{e^t + e^{-t}}$

(i) $\chi(t) = t^2 - 1$,

(ii) $\{-1, 1\}$,

(iii) $\{e^{-t}, e^t\}$,

(iv) $y_h(t) = Ae^{-t} + Be^t,\ A, B \in \mathbb{R},\ t \in \mathbb{R}$.

(viii) $\mathbb{R}$.

(ix)

$$\begin{aligned} c_1' e^t + c_2' e^{-t} &= 0, \\ c_1' e^t - c_2' e^{-t} &= \frac{e^t - e^{-t}}{e^t + e^{-t}}. \end{aligned}$$

(x) $c_1' = \frac{1}{2} e^{-t} \frac{e^t - e^{-t}}{e^t + e^{-t}}$, $c_2' = -\frac{1}{2} e^t \frac{e^t - e^{-t}}{e^t + e^{-t}}$.

(xi) $c_1 = \frac{1}{2} e^{-t} - \arctan(e^{-t})$, $c_2 = -\frac{1}{2} e^t + \arctan(e^t)$.

(xii) $y_p(t) = -e^t \arctan(e^{-t}) + e^{-t} \arctan(e^t) = -\frac{\pi}{2} e^t + (e^{-t} + e^t) \arctan(e^t),\ t \in \mathbb{R}$.

(xiii) $y(t) = y_h(t) + y_p(t) = Ae^{-t} + Be^t + (e^{-t} + e^t) \arctan(e^t),\ A, B \in \mathbb{R},\ t \in \mathbb{R}$.

1.4.2 $y'' - 2y' + y = \frac{e^t}{t^2 + t + 1}$

(i) $\chi(t) = t^2 - 2t + 1$,

(ii) $\{1, 1\}$,

(iii) $\{e^t, te^t\}$,

(iv) $y_h(t) = e^t(A + Bt),\ A, B \in \mathbb{R},\ t \in \mathbb{R}$.

(viii) $\mathbb{R}$.

(ix)

$$\begin{aligned} c_1' e^t + c_2' t e^t &= 0, \\ c_1' e^t + c_2' (t+1) e^t &= \frac{e^t}{t^2 + t + 1}. \end{aligned}$$

(x) $c_1' = -\frac{t}{t^2 + t + 1}$, $c_2' = \frac{1}{t^2 + t + 1}$.

(xi) $c_1 = -\frac{1}{2} \log\left(t^2 + t + 1\right) + \frac{1}{\sqrt{3}} \arctan\left(\frac{2t+1}{\sqrt{3}}\right)$, $c_2 = \frac{2}{\sqrt{3}} \arctan\left(\frac{2t+1}{\sqrt{3}}\right)$.

(xii) $y_p(t) = e^t \left(-\frac{1}{2} \log\left(t^2 + t + 1\right) + \frac{2t+1}{\sqrt{3}} \arctan\left(\frac{2t+1}{\sqrt{3}}\right)\right),\ t \in \mathbb{R}$.

(xiii) $y(t) = y_h(t) + y_p(t) = e^t \left(A + Bt - \frac{1}{2} \log\left(t^2 + t + 1\right) + \frac{2t+1}{\sqrt{3}} \arctan\left(\frac{2t+1}{\sqrt{3}}\right)\right)$, $A, B \in \mathbb{R},\ t \in \mathbb{R}$.

1.4.3 $y'' + y = \tan(t)$

(i) $\chi(t) = t^2 + 1$,

(ii) $\{\pm i\}$,

(iii) $\{\cos(t), \sin(t)\}$,

(iv) $y_h(t) = A\cos(t) + B\sin(t),\ A, B \in \mathbb{R},\ t \in \mathbb{R}$.

(viii) $\mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\}$.

(ix)

$$\begin{aligned} c_1' \cos(t) + c_2' \sin(t) \quad &= 0, \\ -c_1' \sin(t) + c_2' \cos(t) \quad &= \tan(t). \end{aligned}$$

(x) $c_1' = \cos(t) - \frac{1}{\cos(t)}$, $c_2' = \sin(t)$.

(xi) $c_1 = \sin(t) - \int \frac{1}{\cos(t)} dt = \sin(t) - \log\left|\frac{1+\sin(t)}{\cos(t)}\right|$, $c_2 = -\cos(t)$.

(xii) $y_p(t) = -\cos(t)\log\left|\frac{1+\sin(t)}{\cos(t)}\right|$, $t \in \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$, $k \in \mathbb{Z}$.

(xiii) $y(t) = y_h(t) + y_p(t) = \cos(t)\left(A - \log\left|\frac{1+\sin(t)}{\cos(t)}\right|\right) + B\sin(t),\ A, B \in \mathbb{R},\ t \in \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$, $k \in \mathbb{Z}$.

1.4.4 $y'' - 2y' + y = \frac{e^t}{\sqrt{4-t^2}}$

(i) $\chi(t) = t^2 - 2t + 1$,

(ii) $\{1, 1\}$,

(iii) $\{e^t, te^t\}$,

(iv) $y_h(t) = e^t(A + Bt),\ A, B \in \mathbb{R},\ t \in \mathbb{R}$.

(viii) $(-2, 2)$.

(ix)

$$\begin{aligned} c_1' e^t + c_2' t e^t \quad &= 0, \\ c_1' e^t + c_2'(t+1)e^t \quad &= \frac{e^t}{\sqrt{4-t^2}}. \end{aligned}$$

(x) $c_1' = -\frac{t}{\sqrt{4-t^2}}$, $c_2' = \frac{1}{\sqrt{4-t^2}}$.

(xi) $c_1 = \sqrt{4-t^2}$, $c_2 = \arcsin\left(\frac{t}{2}\right)$.

(xii) $y_p(t) = e^t\left(\sqrt{4-t^2} + t\arcsin\left(\frac{t}{2}\right)\right)$, $t \in (-2, 2)$.

(xiii) $y(t) = y_h(t) + y_p(t) = e^t\left(A + Bt + \sqrt{4-t^2} + t\arcsin\left(\frac{t}{2}\right)\right),\ A, B \in \mathbb{R},\ t \in (-2, 2)$.

1.4.5 $y'' - 3y' + 2y = \frac{e^{2t}}{\sqrt{1-e^{2t}}}$

(i) $\chi(t) = t^2 - 3t + 2$,

(ii) $\{1, 2\}$,

(iii) $\{e^t, e^{2t}\}$,

(iv) $y_h(t) = ae^t + be^{2t}$, $a, b \in \mathbb{R}$, $t \in \mathbb{R}$.

(viii) $\mathbb{R}^-$.

(ix)

$$\begin{aligned} c_1' e^t + c_2' e^{2t} &= 0, \\ c_1' e^t + 2c_2' e^{2t} &= \frac{e^{2t}}{\sqrt{1-e^{2t}}}. \end{aligned}$$

(x) $c_1' = -\frac{e^t}{\sqrt{1-e^{2t}}}$, $c_2' = \frac{1}{\sqrt{1-e^{2t}}}$.

(xi) $c_1 = \arccos(e^t)$, $c_2 = t - \log\left(1 + \sqrt{1-e^{2t}}\right)$.

(xii) $y_p(t) = e^t \arccos(e^t) + e^{2t}\left(t - \log\left(1 + \sqrt{1-e^{2t}}\right)\right)$, $t \in \mathbb{R}^-$.

(xiii) $y(t) = y_h(t) + y_p(t) = e^t\left(a + \arccos(e^t)\right) + e^{2t}\left(b + t - \log\left(1 + \sqrt{1-e^{2t}}\right)\right)$, $A, B \in \mathbb{R}$, $t \in \mathbb{R}^-$.